TIGHT CLOSURE AND PLUS CLOSURE IN DIMENSION TWO

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Abstract. We prove that the tight closure and the graded plus closure of a homogeneous ideal coincide for a two-dimensional $\mathbb{N}$-graded domain of finite type over the algebraic closure of a finite field. This answers in this case a "tantalizing question" of Hochster.

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Introduction

One of the basic open questions in tight closure theory is the problem whether the tight closure of an ideal I in a Noetherian domain R of positive characteristic p is the same as its plus closure. Hochster calls this a tantalizing question [6, Remark to Theorem 3.1]. A positive answer to this question would also give a positive answer to the localization problem.

The tight closure of an ideal $I = (f_1, \ldots, f_n) \subseteq R$ is the ideal given by

$$I^* = \{f \in R : \exists c \neq 0 \text{ such that } cf^q \in (f_1^q, \ldots, f_n^q) \text{ for all powers } q = p^e\}\,.$$

The plus closure of I is just $I^+ = R \cap IR^+$, where R^+ is the integral closure of R in an algebraic closure of the quotient field $Q(R)$. An element f belongs to this plus closure if and only if there exists a finite extension $R \subseteq S$ such that $f \in IS$. The inclusion $I^+ \subseteq I^*$ is easy, see [8, Theorem 1.7].

K. Smith proved in [15] that the answer to this question is yes for parameter ideals in an excellent local domain. Recall that d elements in a local ring $(R, \mathfrak{m})$ of dimension d are called parameters if they generate an ideal primary to the maximal ideal $\mathfrak{m}$. Parameter ideals have the advantage that one may consider the tight closure question whether $f \in (f_1, \ldots, f_d)^*$ holds as a problem about the Čech cohomology class $\frac{f}{f_1 \cdots f_d} \in H^d_{\mathfrak{m}}(R)$.

The general question remained open even for homogeneous R_+-primary ideals in a two-dimensional normal standard-graded domain. After a long period without substantial progress it was shown in [4] that the question has a positive answer for homogeneous primary ideals in affine cones over an elliptic curve, such as $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$, itself a prominent example in tight closure theory. This result rests on a geometric interpretation of tight closure theory in terms of vector bundles and uses the classification of vector bundles on elliptic curves due to M. Atiyah.

In this paper we show that $I^* = I^+$ (in fact we even show that $I^* = I^{+\mathrm{gr}}$, the graded plus closure) holds for a homogeneous ideal in a two-dimensional $\mathbb{N}$-graded domain R of finite type over the algebraic closure of a finite field (Theorem 4.2). This last finiteness condition is not necessary in the elliptic case, but it is crucial for our proof for higher genus. The point is that due to a result of H. Lange and U. Stuhler [11] a strongly semistable bundle (see below for the definition) of degree 0 on a smooth projective curve Y defined over a finite field can be trivialized by pulling it back along a finite mapping $Y' \to Y$.

The other main ingredient of our proof is –beside the geometric interpretation of tight closure– a recent theorem of A. Langer. He shows for a locally free sheaf on a smooth projective variety that the Harder-Narasimhan filtration of some Frobenius pull-back has strongly semistable quotients [12, Theorem 2.7]. This allows us to do induction on this strong Harder-Narasimhan filtration and to obtain for the tight closure and the plus closure (or rather for their geometric counterparts) the same numerical criterion.

The paper is organized as follows. In Section 1 we describe briefly the geometric interpretation and the two mentioned results which we will use in the following.

Sections 2 and 3 deal with the geometric setting which arises from tight closure: a locally free sheaf $\mathcal{S}$ on a smooth projective curve Y and a geometric torsor $T \to Y$ corresponding to a cohomology class $c \in H^1(Y, \mathcal{S})$. This setting is of independent interest and no knowledge of tight closure is required.

Section 2 establishes a numerical criterion for the affineness of such a torsor on a curve in every characteristic (Theorem 2.3). Section 3 establishes the same numerical criterion for the existence of a projective curve inside the torsor under the condition that the curve is defined over a finite field (Theorem 3.2).

In Section 4 we derive the consequences from our results to tight closure. The equality $I^* = I^{+\mathrm{gr}}$ for R_+-primary homogeneous ideals in the normal standard-graded case follows immediately (Theorem 4.1) and in Theorem 4.2 we remove the conditions primary, normal and standard-graded.

1. Preliminaries

We describe briefly the geometric setting of our approach to tight closure and plus closure (see [2] for details). Let R denote a normal standard-graded domain over an algebraically closed field K (of any characteristic) and let $f_1, \ldots, f_n$ denote homogeneous generators of an R_+-primary ideal of degrees $d_1, \ldots, d_n$. These data give rise to the short exact sequence of locally free

sheaves on $Y = \operatorname{Proj} R$,

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(m - d_i) \xrightarrow{f_i} \mathcal{O}(m) \longrightarrow 0\,.$$

Another homogeneous element $f \in R$ of degree m yields via the connecting homomorphism the cohomology class $c = \delta(f) \in H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(m))$. Such a class $c \in H^1(Y, \mathcal{S})$ of a locally free sheaf $\mathcal{S}$ yields a geometric $\mathcal{S}$-torsor $T \to Y$, that is an affine-linear bundle on which $\mathcal{S}$ acts.

With this setting, the containment $f \in (f_1, \ldots, f_n)^*$ is in positive characteristic equivalent to the property that the cohomological dimension of the torsor T equals $d = \dim Y = \dim R - 1$ [2, Proposition 3.9]. If R has dimension two, this just means that T is not an affine scheme.

The graded plus closure of an ideal, denoted $I^{+\mathrm{gr}}$, consists of the elements such that there exists a finite homogeneous extension $R \subseteq S$ (of graded domains) such that $f \in IS$. In terms of our geometric setting, the containment $f \in (f_1, \ldots, f_n)^{+\mathrm{gr}}$ is equivalent to the geometric property that the torsor T contains a closed projective subvariety of dimension d [2, Lemma 3.10]. This is equivalent to the property that there exists a finite dominant mapping $\varphi : Y' \to Y$ of a projective normal variety Y' such that $\varphi^*(c) = 0$ in $H^1(Y', \varphi^*(\mathcal{S}))$.

We restrict now to the case of a two-dimensional normal domain R so that Y is a smooth projective curve. A crucial notion to study the affineness of a torsor T given by a cohomology class $c \in H^1(Y, \mathcal{S})$ is the semistability of $\mathcal{S}$. Recall that a locally free sheaf $\mathcal{S}$ is called semistable if $\mu(\mathcal{T}) \leq \mu(\mathcal{S})$ holds for every subbundle $\mathcal{T} \subseteq \mathcal{S}$, where $\mu(\mathcal{T}) = \deg(\mathcal{T}) / \operatorname{rk}(\mathcal{T})$ denotes the slope of a bundle (see [9] for background of this notion).

In positive characteristic, the pull-back of a semistable bundle under the absolute Frobenius $F : Y \to Y$ is in general not semistable. However, if it stays semistable for every Frobenius power, then the bundle is called strongly semistable, a notion introduced by Miyaoka in [14]. Moreover, a strongly semistable bundle stays semistable under every finite mapping $\varphi : Y' \to Y$ [14, Proposition 5.1].

A useful characterization for a strongly semistable bundle on a curve defined over a finite field is given in [11]. In this paper, H. Lange and U. Stuhler study GL-representations of the algebraic fundamental group in positive characteristic and stability properties of the corresponding vector bundles. One of their results says that a vector bundle $\mathcal{S}$ becomes trivial on an étale covering (and corresponds then to a continuous representation of the algebraic fundamental group $\pi_1(Y) \to \mathrm{GL}(r, K)$) if and only if some Frobenius pull-back of $\mathcal{S}$ is isomorphic to $\mathcal{S}$ (which is only possible for $\deg(\mathcal{S}) = 0$). For our purpose the following result is more important.

Theorem 1.1. *Let K denote the algebraic closure of a finite field of characteristic p and let Y denote a smooth projective curve over K. Let $\mathcal{S}$ denote a locally free sheaf on Y of degree 0. Then the following are equivalent.*

(i) *$\mathcal{S}$ trivializes under a finite mapping $Y' \to Y$.*
(ii) *$\mathcal{S}$ is strongly semistable.*
(iii) *There exist numbers $e' > e$ such that $F^{e'*}(\mathcal{S}) \cong F^{e*}(\mathcal{S})$.*
(iv) *There exists $Y' \xrightarrow{\varphi} Y \xrightarrow{F^e} Y$, where φ is étale and where the pull-back $(F^e \circ \varphi)^*(\mathcal{S})$ is trivial.*

Proof. This is proven in [11, Satz 1.9, Korollar 1.6 and Satz 1.4]. $\square$

Remark 1.2. The pull-back of a syzygy bundle $\mathrm{Syz}(f_1, \ldots, f_n)(m)$ under the e-th Frobenius morphism is given by $\mathrm{Syz}(f_1^q, \ldots, f_n^q)(qm)$, where $q = p^e$. So the condition (iii) in Theorem 1.1 translates to $\mathrm{Syz}(f_1^q, \ldots, f_n^q)(qm) \cong \mathrm{Syz}(f_1^{q'}, \ldots, f_n^{q'})(q'm)$. If moreover $\mathrm{Syz}(f_1^q, \ldots, f_n^q)(qm) \cong \mathrm{Syz}(f_1, \ldots, f_n)(m)$ for some $q = p^e$, then there exists even an étale covering Y' of Y such that this syzygy bundle becomes trivial.

For every locally free sheaf $\mathcal{S}$ on the smooth projective curve Y there exists the so-called Harder-Narasimhan filtration $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \mathcal{S}$. This filtration is unique and has the property that the quotients $\mathcal{S}_{i+1}/\mathcal{S}_i$ are semistable and $\mu(\mathcal{S}_i/\mathcal{S}_{i-1}) > \mu(\mathcal{S}_{i+1}/\mathcal{S}_i)$ holds true. The number $\mu(\mathcal{S}_1) = \mu_{\max}(\mathcal{S})$ is called the maximal slope of $\mathcal{S}$ and $\mu(\mathcal{S}/\mathcal{S}_{t-1}) = \mu_{\min}(\mathcal{S})$ the minimal slope of $\mathcal{S}$.

The pull-back of the Harder-Narasimhan filtration under the Frobenius yields in general not the Harder-Narasimhan filtration of $F^*(\mathcal{S})$. However, a recent result of A. Langer [12, Theorem 2.7] shows that there exists a Frobenius power F^e such that the quotients in the Harder-Narasimhan filtration of the pull-back $F^{e*}(\mathcal{S})$ are all strongly semistable. We call such a filtration the strong Harder-Narasimhan filtration.

This strong Harder-Narasimhan filtration is useful for us in several respects. It allows us to reduce some questions to the strongly semistable case, where a numerical characterization for tight closure was given in [3, Theorem 8.4].

It also implies that the number

$$\bar{\mu}_{\min}(\mathcal{S}) = \lim_{e \in \mathbb{N}} \frac{\mu_{\min}(F^{e*}(\mathcal{S}))}{p^e} = \min\{\frac{\mu_{\min}(\varphi^*(\mathcal{S}))}{\deg(\varphi)} : \ \varphi : Y' \to Y \text{ finite}\}$$

is a rational number and that it is obtained for some e. This improves also the slope criterion for ample vector bundles: $\mathcal{S}$ is ample if and only if $\bar{\mu}_{\min}(\mathcal{S}) > 0$ [3, Theorem 2.3], and this is now equivalent to $\mu_{\min}(F^{e*}\mathcal{S}) > 0$ for all $e \in \mathbb{N}$.

Furthermore, if $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \mathcal{S}$ is the strong Harder-Narasimhan filtration, then we may look for the maximal i such that $\mu(\mathcal{S}_i/\mathcal{S}_{i-1}) \geq 0$ (if $\mu(\mathcal{S}_j/\mathcal{S}_{j-1}) < 0$ for all $j = 1, \ldots, t$, then set $i = 0$ and $S_0 = S_{-1} = 0$ and if $\mu(\mathcal{S}_j/\mathcal{S}_{j-1}) \geq 0$ for all $j = 1, \ldots, t$, then set $i = t+1$ and $\mathcal{S}_{t+1} = \mathcal{S}_t = \mathcal{S}$). It

will be crucial to consider the short exact sequence $0 \to \mathcal{S}_i \to \mathcal{S} \to \mathcal{S}/\mathcal{S}_i = \mathcal{Q} \to 0$. Of course, $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_i$ is the strong Harder-Narasimhan filtration of $\mathcal{S}_i$ and $\mathcal{S}_{i+1}/\mathcal{S}_i \subset \mathcal{S}_{i+2}/\mathcal{S}_i \subset \ldots \subset \mathcal{S}/\mathcal{S}_i = \mathcal{Q}$ is the strong Harder-Narasimhan filtration of $\mathcal{Q}$. Therefore $\bar{\mu}_{\min}(\mathcal{S}_i) \geq 0$ and $\bar{\mu}_{\max}(\mathcal{Q}) < 0$. Hence $\bar{\mu}_{\min}(\mathcal{Q}^\vee) > 0$ and the dual $\mathcal{Q}^\vee$ is an ample vector bundle.

2. Slope criteria for the affineness of torsors

Let Y be a smooth projective curve over an algebraically closed field K of arbitrary characteristic. Let $\mathcal{S}$ denote a locally free sheaf on Y. A cohomology class $c \in H^1(Y, \mathcal{S})$ corresponds to a geometric torsor $T \to Y$. This is an affine-linear bundle on which $\mathcal{S}$ acts by translations. A natural realization of T is obtained as follows: since $H^1(Y, \mathcal{S}) \cong \operatorname{Ext}(\mathcal{O}_Y, \mathcal{S})$, the class yields an extension $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$ which gives a projective embedding $\mathbb{P}(\mathcal{S}^\vee) \hookrightarrow \mathbb{P}(\mathcal{S}'^\vee)$. Then $T = \mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$.

We are interested in the global properties of such a torsor $T \to Y$ corresponding to $c \in H^1(Y, \mathcal{S})$. In this section we give a numerical criterion in terms of the strong Harder-Narasimhan filtration in order to decide whether such a torsor is an affine scheme or not.

Proposition 2.1. *Let Y denote a smooth projective curve over an algebraically closed field K. Let $\mathcal{S}$ denote a locally free sheaf and let $c \in H^1(Y, \mathcal{S})$ be a cohomology class with corresponding torsor $T \to Y$. If $\mathcal{S}$ is strongly semistable, then the following are equivalent.*

(i) *The torsor T is an affine scheme.*
(ii) *$\bar{\mu}_{\max}(\mathcal{S}) < 0$ and $c \neq 0$ (in positive characteristic $F^{e*}(c) \neq 0$ for all Frobenius powers F^e).*
(iii) *The dual extension $\mathcal{S}'^\vee$ given by c is an ample bundle.*

The implications (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (i) *hold for any locally free sheaf $\mathcal{S}$.*

Proof. (i) $\Rightarrow$ (ii). Since T is affine, we have $c \neq 0$, for otherwise T would be trivial and contain projective curves. Since the pull-back $T' = T \times_Y Y'$ for $Y' \to Y$ finite is again affine, we get also $F^{e*}(c) \neq 0$ in positive characteristic. The affineness of T implies in general $\bar{\mu}_{\min}(\mathcal{S}) < 0$ due to [3, Theorem 4.4]. But if $\mathcal{S}$ is strongly semistable, then $\bar{\mu}_{\min}(\mathcal{S}) = \bar{\mu}_{\max}(\mathcal{S}) < 0$.

(ii) $\Rightarrow$ (iii). The slope condition yields $\bar{\mu}_{\min}(\mathcal{S}^\vee) > 0$ for the dual bundle $\mathcal{S}^\vee$. Hence $\mathcal{S}^\vee$ is an ample vector bundle (see the discussion at the end of the previous section). Since $\mathcal{S}'^\vee$ is a non-trivial extension, it follows from [5, Proposition 2.2] that all quotients of $\mathcal{S}'^\vee$ have positive degree. Since this is true for every Frobenius pull-back, we have again $\bar{\mu}_{\min}(\mathcal{S}'^\vee) > 0$ and so $\mathcal{S}'^\vee$ is also ample.

(iii) $\Rightarrow$ (i). The ampleness of $\mathcal{S}'^\vee$ is by definition the ampleness of the divisor $\mathbb{P}(\mathcal{S}^\vee) \subset \mathbb{P}(\mathcal{S}'^\vee)$. So the complement of $\mathbb{P}(\mathcal{S}^\vee)$ is an affine scheme.

Finally we prove (iii) $\Rightarrow$ (ii) without the condition that $\mathcal{S}$ is strongly semistable. The surjection $\mathcal{S}'^{\vee} \to \mathcal{S}^{\vee} \to 0$ shows that $\mathcal{S}^{\vee}$ is also ample; hence $\bar{\mu}_{\min}(\mathcal{S}^{\vee}) > 0$ and therefore $\bar{\mu}_{\max}(\mathcal{S}) < 0$. If $F^{e*}(c) = 0$, then $F^{e*}(\mathcal{S}'^{\vee}) \cong \mathcal{S}^{\vee} \oplus \mathcal{O}_Y$ would be not ample. □

Remark 2.2. It may indeed happen that a class $0 \neq c \in H^1(Y, \mathcal{S})$ for $\mathcal{S}$ of negative degree (even for invertible $\mathcal{S}$) becomes zero under a Frobenius power. But this is an exception. In the case of a relative curve over $\operatorname{Spec} \mathbb{Z}$, this may happen only for finitely many prime numbers. For the condition $c \neq 0$ implies ampleness over the generic curve of characteristic zero, and this holds then almost everywhere, since ampleness is an open property.

We look now at the affineness of the torsor T for an arbitrary locally free sheaf $\mathcal{S}$. The crucial point is to look at the strong Harder-Narasimhan filtration of $\mathcal{S}$ (in characteristic zero this is just the usual Harder-Narasimhan filtration, and replace the Frobenius by the identity). Note that the property of being affine is preserved under a finite mapping. So in positive characteristic we often apply an absolute Frobenius morphism $F : Y \to Y$ to pull back the whole situation.

Theorem 2.3. *Let $\mathcal{S}$ denote a locally free sheaf on a smooth projective curve Y over an algebraically closed field K, let $c \in H^1(Y, \mathcal{S})$ with corresponding torsor $T \to Y$. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t$ be the strong Harder-Narasimhan filtration of $F^{e*}(\mathcal{S})$ on Y. Choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0 (we set $i = -1$ and $\mathcal{S}_{-1} = 0$ and $i = t + 1$ and $\mathcal{S}_{t+1} = F^{e*}(\mathcal{S})$ in the extremal cases). Set $0 \to \mathcal{S}_i \to F^{e*}(\mathcal{S}) \to F^{e*}(\mathcal{S})/\mathcal{S}_i = \mathcal{Q} \to 0$. Then T is affine if and only if the image of $F^{e*}(c)$ in $H^1(Y, \mathcal{Q})$ is non-zero (in positive characteristic non-zero for all Frobenius powers).*

Proof. Let $c' \in H^1(Y, \mathcal{Q})$ denote the image of $F^{e*}(c)$ and suppose first that $c' \neq 0$. By construction, $\mathcal{Q}$ is a locally free sheaf $\neq 0$ with $\bar{\mu}_{\max}(\mathcal{Q}) < 0$. So its dual sheaf $\mathcal{Q}^{\vee}$ is an ample vector bundle and the same holds true for the extension $\mathcal{Q}'^{\vee}$ defined by c' as in the proof of Proposition 2.1. Therefore the torsor T' corresponding to c' is affine by Proposition 2.1. There exists an affine morphism $F^{e*}(T) \to T'$ induced by $F^{e*}(\mathcal{S}) \to \mathcal{Q}$ [4, Lemma 3.1], hence $F^{e*}(T)$ and then also T itself is affine.

On the other hand, suppose that some Frobenius pull-back of the image c' is 0. Then we may assume $c' = 0$ and hence $F^{e*}(c)$ stems from a cohomology class $\tilde{c} \in H^1(Y, \mathcal{S}_i)$, where $\bar{\mu}_{\min}(\mathcal{S}_i) \geq 0$ (including the case $\mathcal{S}_i = \mathcal{S}_{-1} = 0$). It is enough to show that the affine-linear bundle $\tilde{T}$ corresponding to $\tilde{c}$ is not affine. But this was proven in [3, Theorem 4.4]. □

3. Slope criteria for the trivialization of a cohomology class

In this section we suppose that K is the algebraic closure of a finite field of positive characteristic p, $K = \overline{\mathbb{F}}_p$. We shall now prove for a smooth projective

curve Y defined over K that the criterion of the last section for the affineness of a torsor holds also for the property that there does not exist any projective curve inside the torsor. This property itself is equivalent to the fact that the cohomolgy class does not get trivial under any finite mapping $\varphi : Y' \to Y$. We first do the case of a strongly semistable bundle.

Theorem 3.1. *Let Y denote a smooth projective curve over the algebraic closure of a finite field. Let $\mathcal{S}$ denote a strongly semistable locally free sheaf of degree ≥ 0 and let $c \in H^1(Y, \mathcal{S})$ denote a cohomology class. Then c trivializes under a finite mapping $Y' \to Y$.*

Proof. Suppose first that the degree of $\mathcal{S}$ is positive. Then $\mathcal{S}$ is an ample bundle and so it is also cohomologically p-ample in the sense of [10] (see [13] or [1]), that is, for every coherent sheaf $\mathcal{F}$ we have $H^i(Y, \mathcal{F} \otimes F^{e*}(\mathcal{S})) = 0$ for $i \geq 1$ and e large enough. This implies in particular that some Frobenius power of the class c is trivial.

So suppose that $\deg(\mathcal{S}) = 0$. Since K is supposed to be the closure of a finite field, every object we encounter is in fact defined already over a finite field. Since $\mathcal{S}$ is strongly semistable, there exists by the result of Lange and Stuhler (see Theorem 1.1 above) a finite mapping $\varphi : Y' \to Y$ (which is an étale mapping followed by a Frobenius power) such that the pull-back is trivial, hence $\varphi^*(\mathcal{S}) = \mathcal{O}_{Y'}^r$. Thus we may assume that $\mathcal{S}$ is in fact a trivial bundle. We may deal with the components of $c \in H^1(Y, \mathcal{O}_Y^r)$ separately, so we may even assume that $\mathcal{S} = \mathcal{O}_Y$. But for the structure sheaf this was proven in [2, Proposition 8.1] and also follows from [16, Proposition 3.3]. $\square$

Theorem 3.2. *Let Y denote a smooth projective curve over the algebraic closure of a finite field. Let $\mathcal{S}$ denote a locally free sheaf on Y and let $c \in H^1(Y, \mathcal{S})$ with corresponding torsor $T \to Y$. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t$ be the strong Harder-Narasimhan filtration of $F^{e*}(\mathcal{S})$ on Y. Choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0. Let $0 \to \mathcal{S}_i \to F^{e*}(\mathcal{S}) \to F^{e*}(\mathcal{S})/\mathcal{S}_i = \mathcal{Q} \to 0$. Then the following are equivalent:*

(i) *The $\mathcal{S}$-torsor T contains a projective curve.*
(ii) *There exists a smooth projective curve Y' and a finite mapping $\varphi : Y' \to Y$ such that $\varphi^*(c) = 0$.*
(iii) *Some Frobenius power of the image of $F^{e*}(c)$ in $H^1(Y, \mathcal{Q})$ is zero.*

Proof. The equivalence of (i) and (ii) is clear, since a torsor is trivial if and only if it has a section. Suppose that (iii) holds. We may assume that the image of $F^{e*}(c)$ is 0 inside $H^1(Y, \mathcal{Q})$. Then $F^{e*}(c)$ stems from a class $c_i \in H^1(Y, \mathcal{S}_i)$. The filtration $0 \subset S_1 \subset \ldots \subset S_{i-1} \subset S_i$ is such that all quotients are strongly semistable with $\deg(\mathcal{S}_j/\mathcal{S}_{j-1}) \geq 0$ for $j = 1, \ldots, i$. We look at the short exact sequence $0 \to \mathcal{S}_{i-1} \to \mathcal{S}_i \to \mathcal{S}_i/\mathcal{S}_{i-1} \to 0$. Since the sheaf on the right is strongly semistable of non-negative degree, it follows from Theorem 3.1 that the image c_i' of c_i in $H^1(Y, \mathcal{S}_i/\mathcal{S}_{i-1})$ vanishes under a finite mapping

$\varphi : Y' \to Y$. Then $\varphi^*(c_i)$ stems from a class $c_{i-1} \in H^1(Y', \varphi^*(\mathcal{S}_{i-1}))$. Going on like this inductively we find a finite mapping such that the pull-back of c is zero.

Condition (i) implies that T is not an affine scheme. So (i) $\Rightarrow$ (iii) follows from Theorem 2.3. □

Bringing the results of this and the last section together we deduce the following corollary, which gives a geometric criterion for the cohomological property of being affine.

Theorem 3.3. *Let Y denote a smooth projective curve over the algebraic closure of a finite field. Let $\mathcal{S}$ denote a locally free sheaf on Y and let $c \in H^1(Y, \mathcal{S})$ with corresponding geometric torsor $T \to Y$. Then T is an affine scheme if and only if it does not contain any projective curve.*

Proof. This follows from Theorem 2.3 and Theorem 3.2, since for both properties the same numerical criterion holds. □

4. Plus closure and tight closure

We come now back to tight closure. The results in the previous sections give at once the identity $I^* = I^+$ for homogeneous primary ideals in a normal standard-graded two-dimensional domain.

Theorem 4.1. *Suppose that K is the algebraic closure of a finite field. Let R denote a standard-graded two-dimensional normal domain over K. Then for every homogeneous R_+-primary ideal I we have the identities $I^* = I^{+\mathrm{gr}} = I^+$.*

Proof. The inclusions $I^{+\mathrm{gr}} \subseteq I^+ \subseteq I^*$ are clear, so it is enough to show that $I^* \subseteq I^{+\mathrm{gr}}$. Due to [7, Theorem 4.2] we only have to consider homogeneous elements. Let $I = (f_1, \ldots, f_n)$ be given by homogeneous ideal generators. These give rise to the locally free syzygy bundle $\mathrm{Syz}(f_1, \ldots, f_n)(m)$ on the smooth projective curve $Y = \mathrm{Proj}\, R$. A homogeneous element f of degree m induces a cohomology class $c = \delta(f) \in H^1(Y, \mathrm{Syz}(f_1, \ldots, f_n)(m))$ with corresponding torsor $T \to Y$. The containment of f in the graded plus closure, $f \in I^{+\mathrm{gr}}$, is equivalent to the existence of a projective curve inside T. The containment in the tight closure, $f \in I^*$, is equivalent to the non-affineness of T. So the result follows from Theorem 3.3. □

Theorem 4.2. *Suppose that K is the algebraic closure of a finite field. Let R denote an $\mathbb{N}$-graded two-dimensional domain of finte type over K. Then for every homogeneous ideal I we have the identities*

$$I^* = I^{+\mathrm{gr}} = I^+ .$$

Proof. We first reduce the identity to the primary case. Let $I = (f_1, \ldots, f_n)$ denote a homogeneous ideal and suppose $f \in I^*$, f homogeneous of degree

m. Then for every $k \in \mathbb{N}$ we have $f \in (I + R_{\geq k})^*$. Since these are R_+-primary ideals, we have also $f \in (I + R_{\geq k})^{+\mathrm{gr}}$. This means that for every k we have a finite, homogeneous (degree-preserving) extension $R \to S$, where S is another graded domain, with $f \in (I + R_{\geq k})S$. This means that we have an equation $f = \sum_{i=1}^n s_i f_i + \sum_j t_j g_j$, where $s_i, t_j \in S$ and $g_j \in R_{\geq k}$. We may assume that everything is homogeneous, hence for $k > m$ we get $t_j = 0$ and therefore $f \in IS$.

Now suppose that R is a two-dimensional $\mathbb{N}$-graded domain R of finite type over the algebraic closure K of a finite field. Write $R = K[T_1, \ldots, T_k]/\mathfrak{a}$ with $\deg(T_i) = e_i$, and look at the homogeneous ring homomorphism

$$K[T_1, \ldots, T_k] \to K[U_1, \ldots, U_k],\ T_i \mapsto U^{e_i}$$

and the induced mapping $R \to K[U_1, \ldots, U_k]/\mathfrak{a}K[U_1, \ldots, U_k] =: R'$. R' is now a standard-graded two-dimensional K-algebra finite over R. We may mod out a homogeneous minimal prime ideal of R' and normalize to get a finite mapping $R \to S$, where S is now a normal two-dimensional standard-graded domain. From $f \in I^*$ we get $f \in (IS)^*$ and hence due to Theorem 4.1 also $f \in (IS)^{+\mathrm{gr}}$. But then also $f \in I^{+\mathrm{gr}}$. □

Of course it is natural to ask whether Theorems 4.2 and Theorem 3.3 hold without the assumption that K is the algebraic closure of a finite field. A special case of this question is the following problem.

Problem 4.3. Let $\mathcal{L}$ denote an invertible sheaf of degree 0 on a smooth projective curve Y over an algebraically closed field K of positive characteristic. Let $c \in H^1(Y, \mathcal{L})$. Does there exist a finite mapping $\varphi : Y' \to Y$ such that $\varphi^*(c) = 0$?

This is true for the structure sheaf and for every $\mathcal{L}$ which is a torsion point in the Picard group. Therefore it is true in general for the algebraic closure of a finite field.

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